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Geodesic Mappings of Spaces with Affine Connections onto Generalized Symmetric and Ricci-Symmetric Spaces

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Abstract: In the paper, we consider geodesic mappings of spaces with an affine connections onto generalized symmetric and Ricci-symmetric spaces. In particular, we studied in detail geodesic mappings of spaces with an affine connections onto 2-, 3-, and m- (Ricci-) symmetric spaces. These spaces play an important role in the General Theory of Relativity. The main results we obtained were generalized to a case of geodesic mappings of spaces with an affine connection onto (Ricci-) symmetric spaces. The main equations of the mappings were obtained as closed mixed systems of PDEs of the Cauchy type in covariant form. For the systems, we have found the maximum number of essential parameters which the solutions depend on. Any m- (Ricci-) symmetric spaces ($m \geq 1$) are geodesically mapped onto many spaces with an affine connection. We can call these spaces *projectivelly m- (Ricci-) symmetric spaces* and for them there exist above-mentioned nontrivial solutions.

Keywords: geodesic mapping; space with an affine connection; m-symmetric space; m-Ricci-symmetric space

MSC: 53B05; 53B50; 35M10

1. Introduction

The paper is devoted to further study of the theory of geodesic mappings of affinely connected spaces. The theory goes back to the paper [1] by T. Levi–Civita in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note the remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems.

The theory of geodesic mappings was developed by T.Y. Thomas, J.M. Thomas, H. Weyl, P.A. Shirokov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, and others [2–17].

The spaces with covariantly constant curvature tensor (*symmetric spaces*) were considered in 1920 by P.A. Shirokov [5], E. Cartan [18], and A. Lichnerowicz [19], and with covariantly parallel curvature tensor (*recurrent spaces*) [20]. The study of symmetric and recurrent spaces is an extensive part of differential geometry and its applications.

It is well-known that the spaces of constant curvature are symmetric and for them E. Beltrami proved that they admit nontrivial geodesic mappings. In 1954, N.S. Sinyukov [7] began to study geodesic mappings of symmetric, recurrent, and semisymmetric spaces with equiaffine connection onto (pseudo-) Riemannian spaces. Continuation of these studies we can find in the works [21–25], V. Fomin [26], I. Hinterleitner, and J. Mikeš [27]. The above-mentioned results have a negative character in the sense that the space of non-constant curvature does not admit nontrivial geodesic mappings. T. Sakaguchi [28] and V. Domashev, J. Mikeš [29] studied similar tasks for holomorphically projective mappings. In the paper by V.Berezovski et al. [30], it is possible to find the generalized case of geodesic mappings of symmetric spaces.

Later, there were studied more generalized spaces than symmetric and recurrent ones. Generalized symmetric and recurrent spaces were comprehensively studied by V.R. Kaigorodov [31–36] from the point of view of the General Theory of Relativity. The paper [35] is a detailed analysis of this issue; it contains 97 citations. In another direction, symmetric spaces are generalized, for example, in works [37,38].

For geodesic mappings of generalized symmetric and recurrent spaces, such problems were solved by J. Mikeš, V.S. Sobchuk, and others [21–27,38–48]. There are many works devoted to issues of the theory of geodesic mappings, for example [49–58].

The above-mentioned results with proofs are in the works [12,13,15,17].

In our work, we continue the study of geodesic mappings of generalized symmetric spaces with an affine connection.

We suppose that all spaces under consideration are spaces with an affine connections without torsion. In addition, we assume that all geometric objects under consideration are not only continuous but also sufficiently smooth.

2. Basic Concepts of the Theory of Geodesic Mappings of Spaces with Affine Connections

A diffeomorphism between two spaces with an affine connections is an one-to-one differentiable mapping, and the inverse mapping is differentiable too. Among diffeomophisms, there are very important ones which are referred to as geodesic mappings.

Let us suppose that a space A_n with an affine connection ∇ admits a diffeomorphism f onto another space $\bar{A}_n$ with an affine connection $\bar{\nabla}$ and locally the spaces are referred to a common coordinate system x, $x = (x^1, x^2, \dots, x^n)$.

Assume $P = \bar{\nabla} - \nabla$ and a in local coordinate system

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{1}$$

where $\Gamma^h_{ij}(x)$ and $\bar{\Gamma}^h_{ij}(x)$ are components of affine connections ∇ and $\bar{\nabla}$ of the spaces A_n and $\bar{A}_n$, respectively, expressed with respect to the common coordinate system x. The tensor P is called a *deformation tensor* of the connections ∇ and $\bar{\nabla}$ with respect to the mapping f.

A curve $\ell\colon x = x(t)$ in a space A_n with an affine connection ∇ is a *geodesic* when its tangent vector $\lambda(t) = dx(t)/dt$ satisfies the equations

$$\nabla_t \lambda = \rho(t) \cdot \lambda,$$

where ∇_t denotes the covariant derivative along ℓ and $\rho(t)$ is some function.

A diffeomorphism $f\colon A_n \to \bar{A}_n$ is an *geodesic mapping* if any geodesic of A_n is mapped under f onto a geodesic in $\bar{A}_n$.

The most known equations of geodesic mappings are the *Levi–Civita equations*. He has obtained the equations for Riemannian spaces [1]. For the case of affinely connected spaces, the equation was later obtained by H. Weyl [4]. In the paper [59], the authors present alternative proofs for the Levi–Civita equation.

It is known [1–17] that the mapping f of a space A_n onto a space $\bar{A}_n$ is geodesic, if and only if in a common coordinate system x the deformation tensor has the form (the Levi–Civita equation)

$$P^h_{ij}(x) = \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i, \tag{2}$$

where δ^h_i is the Kronecker delta and ψ_i is a covector.

A geodesic mapping is called *non-trivial* if $\psi_i \not\equiv 0$. It is obvious that any space A_n with an affine connection admits a non-trivial geodesic mapping onto space $\bar{A}_n$ with an affine connection. However, generally speaking, the similar statement would be wrong for geodesic mappings between Riemannian spaces. In particular, there are classes of Riemannian spaces which do not admit non-trivial geodesic mappings onto other Riemannian spaces.

In the general case, the main equations of geodesic mappings of spaces with an affine connections can not be reduced to closed systems of differential equations of Cauchy-type since the general solutions depend on n arbitrary functions $\psi_i(x)$.

N.S. Sinyukov [8,9] proved that the main equations for geodesic mappings of (pseudo-) Riemannian spaces are equivalent to some linear system of differential equations of Cauchy-type in covariant derivatives.

J. Mikeš and V. Berezovski [50] proved that the main equations for geodesic mapping of space with an affine connection onto a (pseudo-) Riemannian space can also be reduced to a closed system of PDE's of Cauchy type. In the case of geodesic mappings of an equiaffine space onto a (pseudo-) Riemannian space, the main equations are equivalent to some linear system of Cauchy-type in covariant derivatives. This property for all spaces with an affine connection follows from the results by J.M. Thomas [3], see [15,16] that any space with an affine connection is projectivelly equivalent to an equiaffine space.

Refs. [46–48] were devoted to geodesic mappings of spaces with an affine connections onto Ricci-symmetric and 2-Ricci-symmetric spaces. The main equations for the mappings were also obtained as closed systems of PDE's of Cauchy type. A more detailed description of the theory of partial differential equations (PDE's) of the Cauchy type can be found in the books ([15] pp. 100–105) and ([17] pp. 130–134).

It is known [7,12–17] that, in a common coordinate system x, respective to the mapping, the components of the Riemannian tensors R^h_{ijk} and $\bar{R}^h_{ijk}$ of spaces with an affine connections A_n and $\bar{A}_n$, respectively, are in the relation

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{j\alpha} - P^\alpha_{ij}P^h_{k\alpha}. \tag{3}$$

Throughout the paper, the comma denotes the covariant derivative with respect to the connection ∇ of the space A_n. Taking account of (2), from (3), we obtain

$$\bar{R}^h_{ijk} = R^h_{ijk} - \delta^h_j\psi_{i,k} + \delta^h_k\psi_{i,j} - \delta^h_i\psi_{j,k} + \delta^h_i\psi_{k,j} + \delta^h_j\psi_i\psi_k - \delta^h_k\psi_i\psi_j. \tag{4}$$

Contracting the Equation (4) for h and k, we get

$$\bar{R}_{ij} = R_{ij} + n\,\psi_{i,j} - \psi_{j,i} + (1-n)\,\psi_i\psi_j, \tag{5}$$

where R_{ij} and $\bar{R}_{ij}$ are the Ricci tensors of the spaces with an affine connections A_n and $\bar{A}_n$, respectively.

From the Equation (5), it follows that

$$\psi_{i,j} = \psi_i\psi_j + \frac{2}{n^2-1}\,(n\,\bar{R}_{ij} + \bar{R}_{ji} - (n\,R_{ij} + R_{ji})). \tag{6}$$

In particular, Equation (6) was obtained in the papers [46–48].

3. On m-Symmetric Spaces and Ricci m-Symmetric Spaces

As we mentioned earlier, *symmetric spaces* were considered in 1920 by P.A. Shirokov [5], E. Cartan [18], and A. Lichnerowicz [19]. These spaces are characterized by covariantly constant curvature tensor, i.e., $\nabla R = 0$. Their generalizations are *recurrent spaces* studied by H.S. Ruse [20] with covariantly parallel curvature tensor, $\nabla R = \varphi \circ R$. The spaces were generalized in many ways.

One of the most general generalizations are *generalized m-recurrent* (D_n^m), *m-recurrent* (K_n^m) and *m-symmetric* (S_n^m) spaces, which are in turn characterized by relations

$$\begin{aligned}
&\nabla^m R = \Omega \circ R + \Omega^{(1)} \circ \nabla R + \Omega^{(2)} \circ \nabla^2 R + \cdots + \Omega^{(m-1)} \circ \nabla^{m-1} R,\\
&\nabla^m R = \Omega \circ R,\\
&\nabla^m R = 0,
\end{aligned}$$

where $\nabla^m = \nabla(\nabla^{m-1})$, $\Omega (\neq 0)$, $\Omega^{(1)}, \ \ldots \ , \Omega^{(m-1)}$ are tensors.

V.R. Kaigorodov [35] defined these spaces and studied them in detail.

The natural generalizations of these spaces are *generalized m-Ricci-recurrent* ($RicD_n^m$), *m-Ricci-recurrent* ($RicK_n^m$), and *m-Ricci-symmetric* ($RicS_n^m$) spaces, which are in turn characterized by relations

$$\begin{aligned}
&\nabla^m Ric = \Omega \circ Ric + \Omega^{(1)} \circ \nabla Ric + \Omega^{(2)} \circ \nabla^2 Ric + \cdots + \Omega^{(m-1)} \circ \nabla^{m-1} Ric,\\
&\nabla^m Ric = \Omega \circ Ric,\\
&\nabla^m Ric = 0.
\end{aligned}$$

Our work is devoted to the study of the m-symmetric and m-Ricci symmetric spaces. Therefore, we present an example of four-dimensional pseudo-Riemannian m-symmetric spaces, which is not ℓ-symmetric, $\ell = 1, 2, \ldots, m-1$, see ([35] p. 192):

$$ds^2 = -dx^{2^2} - dx^{3^2} + 2dx^4 \cdot [dx^1 + (\alpha_{pq} x^p x^q + \beta_p x^p) dx^4)], \tag{7}$$

where β_p are function on x^4 and α_{pq} are polynoms

$$\alpha_{pq} = \overset{(1)}{a}_{pq} (x^4)^{m-1} + \overset{(2)}{a}_{pq} (x^4)^{m-2} + \cdots + \overset{(m-1)}{a}_{pq} x^4 + \overset{(m)}{a}_{pq},$$

$\overset{(\ell)}{a}_{pq}$ $(\ell = 1, 2, \ldots, m, \ p, q = 2, 3)$ are constants with $\overset{(1)}{a}_{pq} \not\equiv 0$.

We construct an example of 4-dimensional pseudo-Riemannian Ricci m-symmetric spaces which is not Ricci ℓ-symmetric, $\ell = 1, 2, \ldots, m-1$. These spaces are with the above-mentioned metric with function β_p of variable x^4, α_{pq} $(p.q = 2, 3)$ are m times differentiable function of x^4 and $\alpha_{22} + \alpha_{33}$ is the polynom

$$\alpha_{22} + \alpha_{33} = \overset{(1)}{a} (x^4)^{m-1} + \overset{(2)}{a} (x^4)^{m-2} + \cdots + \overset{(m-1)}{a} x^4 + \overset{(m)}{a},$$

where $\overset{(\ell)}{a}$ $(\ell = 1, 2, \ldots, m)$ are constants with $\overset{(1)}{a} \not\equiv 0$.

It is easy to construct more dimensional m-symmetric and m-Ricci symmetric spaces as product spaces of above-mentioned spaces and also trivial spaces which are e.g., spaces of constant curvature.

Recall the main results of geodesic mappings onto m-symmetric and Ricci m-symmetric spaces:

1. N.S. Sinyukov [7]: *If equiaffine symmetric and recurrent spaces admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces $\bar{V}_n$ then $\bar{V}_n$ is the space of constant curvature.*

2. V.V. Fomin [26]: *If symmetric and recurrent spaces with infinity dimension admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces $\bar{V}_n$, then $\bar{V}_n$ is space of constant curvature.*

3. I. Hinterleitner and J. Mikeš [27]: *If equiaffine symmetric and recurrent spaces admit non-trivial geodesic mappings onto (pseudo-) Weyl spaces $\bar{V}_n$, then $\bar{V}_n$ is space of constant curvature.*

4. J. Mikeš [21,23]: *If generalized recurrent spaces D_n^m admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces $\bar{V}_n$, then $\bar{V}_n$ is space of constant curvature.*

5. J. Mikeš [22]: *If Ricci 2-symmetric spaces admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces $\bar{V}_n$, then $\bar{V}_n$ is Einstein space.*

6. J. Mikeš, V.S. Sobchuk [39]: *If Ricci 3-symmetric spaces admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces $\bar{V}_n$, then $\bar{V}_n$ is Einstein space.*

7. V.S. Sobchuk [40]: *If Ricci 4-symmetric spaces admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces $\bar{V}_n$, then $\bar{V}_n$ is Einstein space.*

8. J. Mikeš [23,24]: *If Ricci m-symmetric spaces admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces $\bar{V}_n$, then $\bar{V}_n$ is Einstein space.*

A summary of these results and their proofs contain monographs [12–17].

The above results are "trivial" geodesic mappings in nature, i.e., under the above conditions, the spaces allow only trivial geodesic mapping. On the other hand, spaces with an affine connection, m-symmetric, and m-Ricci-symmetric spaces admit non trivial geodesic mappings onto affine connected spaces $\bar{A}_n$. Our article is devoted to these questions.

4. Geodesic Mappings of Spaces with Affine Connections onto m-Symmetric Spaces

1. We study geodesic mappings f of a space A_n with an affine connection ∇ onto 2-*symmetric space* $\bar{A}_n$ with an affine connection $\bar{\nabla}$, which are characterized by the following condition [35]:

$$\bar{R}^h_{ijk;m\rho} = 0, \tag{8}$$

where the symbol " ; " denotes a covariant derivative with respect to the connection of the space $\bar{A}_n$.

Since

$$\bar{R}^h_{ijk;m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha}$$

take into account (1), it follows that

$$\bar{R}^h_{ijk;m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{9}$$

From Equations (2) and (9), we get

$$\bar{R}^h_{ijk;m} = \bar{R}^h_{ijk,m} + \delta^h_m\psi_\alpha\bar{R}^\alpha_{ijk} - 2\psi_m\bar{R}^h_{ijk} - \psi_j\bar{R}^h_{imk} - \psi_k\bar{R}^h_{ijm} - \psi_i\bar{R}^h_{mjk}. \tag{10}$$

Let us differentiate (10) with respect to x^ρ in the space A_n. Taking into account (6), we get

$$(\bar{R}^h_{ijk;m})_{,\rho} = \bar{R}^h_{ijk,m\rho} + \delta^h_m\psi_\alpha\bar{R}^\alpha_{ijk,\rho} - 2\psi_m\bar{R}^h_{ijk,\rho} - \psi_j\bar{R}^h_{imk,\rho} - \psi_k\bar{R}^h_{ijm,\rho} - \psi_i\bar{R}^h_{mjk,\rho} + B^h_{ijkm\rho}, \tag{11}$$

where

$$\begin{aligned} B^h_{ijkm\rho} \;=\;& \delta^h_m\bar{R}^\alpha_{ijk}\left(\tfrac{1}{n^2-1}\left(n\,\bar{R}_{\alpha\rho} + \bar{R}_{\rho\alpha} - (n\,R_{\alpha\rho} + R_{\rho\alpha})\right) + \psi_\alpha\psi_\rho\right) \\ & -2\bar{R}^h_{ijk}\left(\tfrac{1}{n^2-1}\left(n\,\bar{R}_{m\rho} + \bar{R}_{\rho m} - (n\,R_{m\rho} + R_{\rho m})\right) + \psi_m\psi_\rho\right) \\ & -\bar{R}^h_{imk}\left(\tfrac{1}{n^2-1}\left(n\,\bar{R}_{j\rho} + \bar{R}_{\rho j} - (n\,R_{j\rho} + R_{\rho j})\right) + \psi_j\psi_\rho\right) \\ & -\bar{R}^h_{ijm}\left(\tfrac{1}{n^2-1}\left(n\,\bar{R}_{k\rho} + \bar{R}_{\rho k} - (n\,R_{k\rho} + R_{\rho k})\right) + \psi_k\psi_\rho\right) \\ & -\bar{R}^h_{mjk}\left(\tfrac{1}{n^2-1}\left(n\,\bar{R}_{i\rho} + \bar{R}_{\rho i} - (n\,R_{i\rho} + R_{\rho i})\right) + \psi_i\psi_\rho\right). \end{aligned}$$

From the definition of covariant derivative in consequence of (1), we find

$$(\bar{R}^h_{ijk;m})_{,\rho} = \bar{R}^h_{ijk;m\rho} - P^h_{\alpha\rho}\bar{R}^\alpha_{ijk;m} + P^\alpha_{i\rho}\bar{R}^h_{\alpha jk;m} + P^\alpha_{j\rho}\bar{R}^h_{i\alpha k;m} + P^\alpha_{k\rho}\bar{R}^h_{ij\alpha;m} + P^\alpha_{m\rho}\bar{R}^h_{ijk;\alpha}. \tag{12}$$

Transforming (12) and taking into account (2) and (5), we get

$$(\bar{R}^h_{ijk;m})_{,\rho} = \bar{R}^h_{ijk;m\rho} + C^h_{ijkm\rho}. \tag{13}$$

where

$$\begin{aligned} C^h_{ijkm\rho} \quad = \quad & -\delta^h_\rho \psi_\alpha\, (\bar{R}^\alpha_{ijk,m} + \delta^\alpha_m \psi_\beta \bar{R}^\beta_{ijk} - 2\psi_m \bar{R}^\alpha_{ijk} - \psi_j \bar{R}^\alpha_{imk} - \psi_k \bar{R}^\alpha_{ijm} - \psi_i \bar{R}^\alpha_{mjk}) \\ & +3\psi_\rho\, (\bar{R}^h_{ijk,m} + \delta^h_m \psi_\alpha \bar{R}^\alpha_{ijk} - 2\psi_m \bar{R}^h_{ijk} - \psi_j \bar{R}^h_{imk} - \psi_k \bar{R}^h_{ijm} - \psi_i \bar{R}^h_{mjk}) \\ & +\psi_i\, (\bar{R}^h_{\rho jk,m} + \delta^h_m \psi_\alpha \bar{R}^\alpha_{\rho jk} - 2\psi_m \bar{R}^h_{\rho jk} - \psi_j \bar{R}^h_{\rho mk} - \psi_k \bar{R}^h_{\rho jm} - \psi_\rho \bar{R}^h_{mjk}) \\ & +\psi_j\, (\bar{R}^h_{i\rho k,m} + \delta^h_m \psi_\alpha \bar{R}^\alpha_{i\rho k} - 2\psi_m \bar{R}^h_{i\rho k} - \psi_\rho \bar{R}^h_{imk} - \psi_k \bar{R}^h_{i\rho k} - \psi_i \bar{R}^h_{m\rho k}) \\ & +\psi_k\, (\bar{R}^h_{ij\rho,m} + \delta^h_m \psi_\alpha \bar{R}^\alpha_{ij\rho} - 2\psi_m \bar{R}^h_{ij\rho} - \psi_j \bar{R}^h_{im\rho} - \psi_\rho \bar{R}^h_{ijm} - \psi_i \bar{R}^h_{mj\rho}) \\ & +\psi_m\, (\bar{R}^h_{ijk,\rho} + \delta^h_\rho \psi_\alpha \bar{R}^\alpha_{ijk} - 2\psi_\rho \bar{R}^h_{ijk} - \psi_j \bar{R}^h_{i\rho k} - \psi_k \bar{R}^h_{ij\rho} - \psi_i \bar{R}^h_{mjk}). \end{aligned} \tag{14}$$

Let us introduce a tensor $\bar{R}^h_{ijkm}$ defined by

$$\bar{R}^h_{ijk,m} = \bar{R}^h_{ijkm}. \tag{15}$$

In this case, we suppose that in (14) covariant derivatives of the tensor $\bar{R}^h_{ijk}$ with respect to the connection of the space A_n are expressed according to (15).

From (11) and (13), we get

$$\bar{R}^h_{ijk;m\rho} = \bar{R}^h_{ijk,m\rho} + \delta^h_m \psi_\alpha \bar{R}^\alpha_{ijk,\rho} - 2\psi_m \bar{R}^h_{ijk,\rho} - \psi_j \bar{R}^h_{imk,\rho} - \psi_k \bar{R}^h_{ijm,\rho} - \psi_i \bar{R}^h_{mjk,\rho} + B^h_{ijkm\rho} - C^h_{ijkm\rho}. \tag{16}$$

Let us assume that the space $\bar{A}_n$ is 2-symmetric. Hence, from (16), take into account (8) and (15), we find

$$\bar{R}^h_{ijk,m\rho} = 2\psi_m \bar{R}^h_{ijk\rho} + \psi_j \bar{R}^h_{imk\rho} + \psi_k \bar{R}^h_{ijm\rho} + \psi_i \bar{R}^h_{mjk\rho} - \delta^h_m \psi_\alpha \bar{R}^\alpha_{ijk\rho} - B^h_{ijkm\rho} + C^h_{ijkm\rho}. \tag{17}$$

Obviously, in the space A_n, Equations (6), (15) and (17) form a closed mixed system of PDE's of Cauchy type with respect to functions $\psi_i(x)$, $\bar{R}^h_{ijk}(x)$ and $\bar{R}^h_{ijkm}(x)$. The functions $\bar{R}^h_{ijk}(x)$ and $\bar{R}^h_{ijkm}(x)$ must satisfy the algebraic conditions (Ricci and Bianchi identities):

$$\bar{R}^h_{i(jk)} = \bar{R}^h_{(ijk)} = 0, \text{ and also } \bar{R}^h_{i(jk)m} = \bar{R}^h_{(ijk)m} = 0. \tag{18}$$

Hence, we have given the proof.

Theorem 1. *A space A_n with an affine connection admits a geodesic mapping onto a 2-symmetric space $\bar{A}_n$ if and only if the mixed system of differential equations of Cauchy type in covariant derivatives* (6), (15), (17) *and* (18) *has a solution with respect to the functions $\psi_i(x)$, $\bar{R}^h_{ijk}(x)$, and $\bar{R}^h_{ijkm}(x)$.*

Obviously, the number of components of $\psi_i(x)$, $\bar{R}^h_{ijk}(x)$, $\bar{R}^h_{ijkm}(x)$ is $n + n^4 + n^5$. Therefore, a general solution of Cauchy type system (6), (15), (17) and (18) depends on the initial conditions of these components at some point x_0. This means that the solution depends on a finite number of essential parameters. However, from conditions (19), this number of parameters is reduced, and even more so when we take into account the integrability conditions. Estimation of the parameters is in the following corollary.

Corollary 1. *The general solution of the mixed system of Cauchy type* (6), (15), (17) *and* (18) *depends on no more than $n + \frac{1}{2}n(n+1)(n-1)^2$ essential parameters.*

2. Now, we study geodesic mapping of space A_n onto 3-*symmetric space* $\bar{A}_n$, which are characterized by the following conditions [35]:

$$\bar{R}^h_{ijk;m\rho l} = 0. \tag{19}$$

Let us covariantly differentiate (16) with respect to x^l in the space A_n and on the left-hand side express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\bar{A}_n$, using the formula

$$(\bar{R}^h_{ijk;m\rho})_{,l} = \bar{R}^h_{ijk;m\rho l} - P^h_{\alpha l}\bar{R}^\alpha_{ijk;m\rho} + P^\alpha_{il}\bar{R}^h_{\alpha jk;m\rho} + P^\alpha_{jl}\bar{R}^h_{i\alpha k;m\rho} + P^\alpha_{kl}\bar{R}^h_{ij\alpha;m\rho} + P^\alpha_{ml}\bar{R}^h_{ijk;\alpha\rho} + P^\alpha_{\rho l}\bar{R}^h_{ijk;m\alpha}.$$

Let tensor $\bar{R}^h_{ijkm\rho}$ be defined by

$$\bar{R}^h_{ijkm,\rho} = \bar{R}^h_{ijkm\rho}. \tag{20}$$

Let us assume that the space $\bar{A}_n$ is 3-symmetric. Hence, from the obtained equation because of (19), using substitutions and transformations, we find

$$\bar{R}^h_{ijkm\rho,l} = \theta^h_{ijkm\rho l}, \tag{21}$$

where $\theta^h_{ijkm\rho l}$ is some tensor depending on unknown tensors ψ_i, $\bar{R}^h_{ijk}$, $\bar{R}^h_{ijkm}$, $\bar{R}^h_{ijkm\rho}$, and on some tensors, which are assumed to be known.

Obviously, in the space A_n, Equations (6), (15), (20) and (21) form a closed mixed system of PDE's of Cauchy type with respect to functions $\psi_i(x)$, $\bar{R}^h_{ijk}(x)$, $\bar{R}^h_{ijkm}(x)$ and $\bar{R}^h_{ijkm\rho}(x)$. In addition, the algebraic conditions (18) have to be satisfied.

Hence, we have proved

Theorem 2. *A space A_n with an affine connection admits a geodesic mapping onto a* 3*-symmetric space $\bar{A}_n$ if and only if the mixed system of differential equations of Cauchy type in covariant derivatives* (6), (15), (20), (21) *and* (18) *has a solution with respect to the functions $\psi_i(x)$, $\bar{R}^h_{ijk}(x)$, $\bar{R}^h_{ijkm}(x)$ and $\bar{R}^h_{ijkm\rho}(x)$.*

The following parameters estimation follows from the Ricci identity of curvature tensor and its derivatives.

Corollary 2. *The general solution of the mixed system of Cauchy type* (6), (15), (20), (21) *and* (18) *depends on no more than $n + \frac{1}{3}n(n^2+n+1)(n^2-1)$ essential parameters.*

3. Finally, we study geodesic mappings of space A_n onto *m-symmetric space* $\bar{A}_n$, which are characterized by the following condition [35]:

$$\bar{R}^h_{ijk;\rho_1\rho_2\dots\rho_m} = 0. \tag{22}$$

Let us differentiate (21) covariantly $(m-2)$ times with respect to the connection of the space A_n and on the left-hand side express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\bar{A}_n$, using the formula

$$\begin{aligned}(\bar{R}^h_{ijk;\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}})_{,\rho_\tau} \;=\; & \bar{R}^h_{ijk;\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}\rho_\tau} - P^h_{\alpha\rho_\tau}\bar{R}^\alpha_{ijk;\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{i\rho_\tau}\bar{R}^h_{\alpha jk;\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}}\\ & + P^\alpha_{j\rho_\tau}\bar{R}^h_{i\alpha k;\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{k\rho_\tau}\bar{R}^h_{ij\alpha;\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}}\\ & + P^\alpha_{\rho_1\rho_\tau}\bar{R}^h_{ijk;\alpha\dots\rho_{\tau-2}\rho_{\tau-1}} + \cdots + P^\alpha_{\rho_{\tau-1}\rho_\tau}\bar{R}^h_{ijk;\rho_1\dots\rho_{\tau-2}\alpha}.\end{aligned}$$

The formula holds because of (1).

Let us introduce a tensor $\bar{R}^h_{ijk\rho_1\rho_2\rho_3}, \ldots, \bar{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}$ as follows

$$\begin{array}{rcl}\bar{R}^h_{ijk\rho_1\rho_2,\rho_3} & = & \bar{R}^h_{ijk\rho_1\rho_2\rho_3}, \\ & \cdots\cdots\cdots\cdots & \\ \bar{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2},\rho_{m-1}} & = & \bar{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}.\end{array} \tag{23}$$

Let us assume that the space $\bar{A}_n$ is m-symmetric $(m > 3)$. Hence, from the obtained equation because of (22), using substitutions and transformations, taking account of (15), (20), (23), we get

$$\bar{R}^h_{ijk\rho_1\ldots\rho_{m-2}\rho_{m-1},\rho_m} = \theta^h_{ijk\rho_1\ldots\rho_{m-1}\rho_m}, \tag{24}$$

where $\theta^h_{ijk\rho_1\ldots\rho_{m-1}\rho_m}$ is some tensor depending on unknown tensors ψ_i, $\bar{R}^h_{ijk}$, $\bar{R}^h_{ijk\rho_1}$, …, $\bar{R}^h_{ijk\rho_1\ldots\rho_{m-1}}$, and on some tensors, which are assumed to be known.

Obviously, in the space A_n the Equations (6), (15), (20), (23), (24) form a closed mixed system of PDE's of Cauchy type with respect to functions $\psi_i(x)$, $\bar{R}^h_{ijk}(x)$, $\bar{R}^h_{ijk\rho_1}(x), \ldots, \bar{R}^h_{ijk\rho_1\ldots\rho_{m-1}}(x)$. In addition, the algebraic conditions (18) have to be satisfied.

Hence, we have given the proof.

Theorem 3. *A space A_n with an affine connection admits a geodesic mapping onto a m-symmetric space $\bar{A}_n$ if and only if the mixed system of differential equations of Cauchy type in covariant derivatives (6), (15), (18), (20), (23) and (24) has a solution with respect to the functions $\psi_i(x)$, $\bar{R}^h_{ijk}(x)$, $\bar{R}^h_{ijk\rho_1}(x), \ldots, \bar{R}^h_{ijk\rho_1\ldots\rho_{m-1}}(x)$.*

Corollary 3. *The general solution of the mixed system of Cauchy type (6), (15), (18), (20), (23) and (24) depends on no more than $n + \frac{1}{3}n(n^2-1)(1+n+n^2+\cdots+n^{m-1})$ essential parameters.*

5. Geodesic Mappings of Spaces with Affine Connections onto *m*-Ricci-Symmetric Spaces

1. Here, we study geodesic mappings of space A_n onto 2-*Ricci-symmetric space* $\bar{A}_n$, which are characterized by the following condition:

$$\bar{R}_{ij;km} = 0, \tag{25}$$

where $\bar{R}_{ij}$ is the Ricci tensor of $\bar{A}_n$.

Let us contract the Equation (16) for h and k. Because of expressions for the tensors $B^h_{ijkm\rho}$ and $C^h_{ijkm\rho}$, we find

$$\begin{array}{rcl}\bar{R}_{ij;m\rho} & = & \bar{R}_{ij,m\rho} - 3\bar{R}_{ij,(\rho}\psi_{m)} + 3\psi_i\psi_{(\rho}\bar{R}_{m)j} + 6\psi_\rho\psi_m\bar{R}_{ij} + 3\psi_j\bar{R}_{i(\rho}\psi_{m)} - \psi_i\bar{R}_{(m|j|,\rho)} \\ & & -\psi_j\bar{R}_{i(\rho,m)} + \psi_i\psi_j\bar{R}_{(m\rho)} - \frac{1}{n^2-1}\left(2\bar{R}_{ij}\left(n\bar{R}_{m\rho} + \bar{R}_{\rho m} - nR_{m\rho} - R_{m\rho}\right)\right. \\ & & \left.+\bar{R}_{mj}(n\bar{R}_{i\rho} + \bar{R}_{\rho i} - nR_{i\rho} - R_{\rho i}) + \bar{R}_{im}(n\bar{R}_{j\rho} + \bar{R}_{\rho j} - nR_{j\rho} - R_{\rho j})\right),\end{array} \tag{26}$$

where (ij) denotes an operation called symmetrization without division with respect to the indices i and j.

Let us introduce a tensor $\bar{R}_{ijm}$

$$\bar{R}_{ij,m} = \bar{R}_{ijm}. \tag{27}$$

Let us assume that space $\bar{A}_n$ is 2-symmetric. Hence, from (26), take into account (25) and (27), we find

$$\begin{aligned}\bar{R}_{ijm,k} \;=\;& 3\bar{R}_{ij(k}\psi_{m)} - 3\psi_i\psi_{(k}\bar{R}_{m)j} - 6\psi_k\psi_m\bar{R}_{ij} - 3\psi_j\bar{R}_{i(k}\psi_{m)} + \psi_i\bar{R}_{(m|j|k)} \\ & +\psi_j\bar{R}_{i(km)} - \psi_i\psi_j\bar{R}_{(mk)} + \tfrac{1}{n^2-1}\left(2\bar{R}_{ij}\left(n\bar{R}_{mk} + \bar{R}_{km} - nR_{mk} - R_{km}\right)\right. \\ & \left.+\bar{R}_{mj}(n\bar{R}_{ik} + \bar{R}_{ki} - nR_{ik} - R_{ki}) + \bar{R}_{im}(n\bar{R}_{jk} + \bar{R}_{kj} - nR_{jk} - R_{kj})\right).\end{aligned} \tag{28}$$

Obviously, in the space A_n, Equations (6), (27) and (28) form a closed system of PDE's of Cauchy type with respect to functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, and $\bar{R}_{ijk}(x)$.

Hence, we have given the proof.

Theorem 4. *A space A_n with an affine connection admits a geodesic mapping onto a 2-Ricci-symmetric space $\bar{A}_n$ if and only if the closed system of differential equations of Cauchy type in covariant derivatives (6), (27) and (28) has a solution with respect to the functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, and $\bar{R}_{ijk}(x)$.*

Systems (6), (27) and (28) have no more than one solution for initial conditions of components $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ijk}(x)$ at some point x_0. The number of parameters of $\psi_i(x_0)$, $\bar{R}_{ij}(x_0)$ and $\bar{R}_{ijk}(x_0)$ are $n + n^2 + n^3$. Therefore, the following corollary holds.

Corollary 4. *The general solution of the system of Cauchy type (6), (27) and (28) depends on no more than $n + n^2 + n^3$ essential parameters.*

2. Now, we study geodesic mappings of space A_n onto 3-*Ricci-symmetric space* $\bar{V}_n$, which are characterized by the condition:

$$\bar{R}_{ij;kml} = 0. \tag{29}$$

Let us covariantly differentiate (26) with respect to x^l in the space A_n and on the left-hand side express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\bar{A}_n$, using the formula

$$(\bar{R}_{ij;mk})_{,l} = \bar{R}_{ij;mkl} + P^{\alpha}_{il}\bar{R}_{\alpha j;mk} + P^{\alpha}_{jl}\bar{R}_{i\alpha;mk} + P^{\alpha}_{ml}\bar{R}_{ij;\alpha k} + P^{\alpha}_{kl}\bar{R}_{ij;m\alpha}.$$

Using the formulas for transition from the covariant derivatives with respect to the connection of the space $\bar{A}_n$ to the the covariant derivatives with respect to the connection of the space A_n, we find

$$\bar{R}_{ij;mkl} = \bar{R}_{ij,mkl} + \Omega_{ijmkl}, \tag{30}$$

where Ω_{ijmkl} is some tensor, which depends on unknown tensors ψ_i, $\bar{R}_{ij}$, $\bar{R}_{ij,k}$, $\bar{R}_{ij,km}$ and, on some tensors, which are assumed to be known.

Let us introduce a tensor $\bar{R}_{ijmk}$ defined by

$$\bar{R}_{ijm,k} = \bar{R}_{ijmk}. \tag{31}$$

Let us assume that the space $\bar{A}_n$ is 3-symmetric. Hence, from (30), taking into account (27) and (31), we find

$$\bar{R}_{ijmk,l} = -\Omega_{ijmkl}, \tag{32}$$

where the tensor Ω_{ijmkl} depends on the unknown tensors ψ_i, $\bar{R}_{ij}$, $\bar{R}_{ijk}$, and $\bar{R}_{ijkm}$.

Obviously, in the space A_n, Equations (6), (27), (31) and (32) form a closed system of PDE's of Cauchy type with respect to functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ijk}(x)$ and $\bar{R}_{ijkm}(x)$.

Hence, we have proved

Theorem 5. *A space A_n with an affine connection admits a geodesic mapping onto a* 3*-Ricci-symmetric space $\bar{A}_n$ if and only if the system of differential equations of Cauchy type in covariant derivatives* (6)*,* (27)*,* (31) *and* (32) *has a solution with respect to the functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ijk}(x)$, and $\bar{R}_{ijkm}(x)$.*

Corollary 5. *The general solution of the system of Cauchy type* (6)*,* (27)*,* (31) *and* (32) *depends on no more than $n + n^2 + n^3 + n^4$ essential parameters.*

3. Finally, we study geodesic mapping of space V_n onto *m-Ricci-symmetric space* $\bar{V}_n$, which are characterized by the condition:

$$\bar{R}_{ij;\rho_1\rho_2\ldots\rho_m} = 0. \tag{33}$$

It is essentially that 2- and 3-Ricci-symmetric spaces are particular cases of m-Ricci-symmetric spaces.

Let us differentiate (32) covariantly $(m-2)$ times with respect to the connection of the space A_n and on the left-hand side express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\bar{A}_n$, using the formula

$$\begin{aligned}\left(\bar{R}_{ij;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}}\right)_{,\rho_\tau} &= \bar{R}_{ij;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}\rho_\tau} + P^{\alpha}_{i\rho_\tau}\bar{R}_{\alpha j;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + P^{\alpha}_{j\rho_\tau}\bar{R}_{i\alpha;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} \\ &\quad + P^{\alpha}_{\rho_1\rho_\tau}\bar{R}_{ij;\alpha\ldots\rho_{\tau-2}\rho_{\tau-1}} + \cdots + P^{\alpha}_{\tau-1\rho_\tau}\bar{R}_{ij;\rho_1\ldots\rho_{\tau-2}\alpha}.\end{aligned}$$

Let us introduce tensors $\bar{R}_{ijk\rho_1\rho_2\rho_3}, \ldots, \bar{R}_{ij\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}$ as follows:

$$\begin{aligned}\bar{R}_{ij\rho_1\rho_2,\rho_3} &= \bar{R}_{ijk\rho_1\rho_2\rho_3}, \\ &\cdots\cdots\cdots \\ \bar{R}_{ij\rho_1\rho_2\ldots\rho_{m-2},\rho_{m-1}} &= \bar{R}_{ij\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}.\end{aligned} \tag{34}$$

Let us assume that the space $\bar{A}_n$ is m-Ricci-symmetric $(m > 2)$. Hence, from the obtained equation because of (33), using substitutions and transformations, taking into account (27), (31) and (34), we get

$$\bar{R}_{ij\rho_1\ldots\rho_{m-2}\rho_{m-1},\rho_m} = \theta_{ij\rho_1\ldots\rho_{m-1}\rho_m}, \tag{35}$$

where $\theta_{ij\rho_1\ldots\rho_{m-1}\rho_m}$ is some tensor depending on unknown tensors ψ_i, $\bar{R}_{ij}, \ldots, \bar{R}_{ij\rho_1}, \bar{R}_{ij\rho_1\ldots\rho_{m-1}}$, and, on some tensors, which are assumed to be known.

Obviously, in the space A_n, Equations (6), (27), (31), (34) and (35) form a closed system of PDE's of Cauchy type with respect to the functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ij\rho_1}(x), \ldots, \bar{R}_{ij\rho_1\ldots\rho_{m-1}}(x)$.

Hence, we have given the proof.

Theorem 6. *A space A_n with an affine connection admits a geodesic mapping onto a m-symmetric space $\bar{A}_n$ if and only if the system of differential equations of Cauchy type in covariant derivatives* (6)*,* (27)*,* (31)*,* (34) *and* (35) *has a solution with respect to the functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ij\rho_1}(x), \ldots, \bar{R}_{ij\rho_1\ldots\rho_{m-1}}(x)$.*

Corollary 6. *The general solution of the closed system of Cauchy type* (6)*,* (27)*,* (31)*,* (34) *and* (35) *depends on no more than $n + n^2 + \ldots + n^{m+1}$ essential parameters.*

6. Conclusions

The above new results are the generalization of analogical results for geodesic mappings onto symmetric and Ricci-symmetric spaces and for the case of m-symmetric and m-Ricci symmetric spaces $(m > 1)$. The generalization of m-recurrent, m-Ricci-recurrent and also D^m_n and $RicD^m_n$ studied by Kaigorodov probably does not exist in the general case.

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